A Note on the Calculation of the Current Distribution in Lossy Microstrip Structures

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Abstract—**The equations that govern the current distribution in the finite-thickness conductor of a microstrip structure are developed in a rigorous manner. It is shown that the cross-sectional variation of the current density is independent of the field variation along the axis of the microstrip line only for the case when the displacement current in the conductor is negligible compared to the conduction current, a condition easily fulfilled for most practical applications, and the line is operated in the quasi-TEM mode. The validity of various methods proposed recently for conductor loss calculations is discussed on the basis of this analysis.**

I. Introduction

Recently, several methods have been proposed for the calculation of the current distribution over the conductor cross sections in microstrip structures, in order to predict more accurately the associated conductor losses [1]–[6]. Such analyses have been prompted by recent applications of microstrip lines in MMIC's and high speed digital VLSI/ULSI interconnections. Indeed, the conductor dimensions and frequencies of interest in these technologies are such that Wheeler's incremental inductance rule [7] no longer applies, and knowledge of the actual current distribution is required for the accurate evaluation of the frequency-dependent series resistance of the line.

Several of the proposed methods used for the prediction of microstrip current distributions involve two key assumptions [1]–[5]. The first one is that the transverse components of the current can be neglected compared to the longitudinal component in the calculation of the attenuation constant and other transmission line parameters. The second is that the $\exp(-jk_z z)$ dependence of the fields along the axis of the microstrip structure, taken to be parallel to the z axis of the cartesian coordinate system, can be ignored in calculating the current distribution over the cross sections of the conductors. While the first assumption is a valid one for microstrip lines with cross-sectional dimensions small compared to the wavelength of interest, the second one does not seem to have a concrete theoretical justification [5]. This note attempts to provide such a justification.

First, Maxwell's equations with electric and magnetic fields exhibiting an $\exp(-jk_z z)$ dependence are used to develop a reduced set of equations for the transverse field variation over the conductor cross section of a microstrip line. These equations are then used to derive the conditions under which the cross-sectional variation of the current density inside the conductors can be obtained independently of the z dependence of the fields. Finally, the validity of the various methods proposed recently for the prediction of the current distribution in the conductors is examined in light of this analysis.

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II. Mathematical Formulation

Let k_z be the complex propagation constant for a microstrip structure of constant cross section, having its axis parallel to the z axis of a cartesian coordinate system. For wave propagation in the positive z axis, the hybrid-mode fields of the microstrip have the form

$$E(x, y, z) = (\boldsymbol{e}_t(x, y) + \hat{z}e_z(x, y))\exp(-jk_z z), \quad (1)$$

$$H(x, y, z) = (\boldsymbol{h}_t(x, y) + \hat{z}h_z(x, y))\exp(-jk_z z), \quad (2)$$

where $\boldsymbol{e}_t$, $\boldsymbol{h}_t$ denote the transverse part of the fields, and the $\exp(j\omega t)$ time dependence has been suppressed. The objective is to obtain from the general Maxwell's equations a reduced set of equations that govern the cross-sectional variation of the fields inside the conductors of the microstrip line for frequencies such that the displacement current is negligible compared to the conduction current. For such frequencies, $\sigma \gg \omega\epsilon_m$, where σ is the conductivity, and the permittivity ϵ_m of the conductor is roughly the same as that for free space. In addition, $\nabla \cdot \boldsymbol{E} = 0$ inside the conductor, and electric charge can occur only on the surface of the conductor. This point has been discussed in more detail in [8]. Substitution of (1) and (2) in Maxwell's curl equations, with the displacement current term neglected, yields

$$\nabla_t \times \boldsymbol{e}_t = -j\omega\mu h_z \hat{z}, \quad (3)$$

$$-jk_z \hat{z} \times \boldsymbol{e}_t + \nabla_t e_z \times \hat{z} = -j\omega\mu \boldsymbol{h}_t, \quad (4)$$

$$\nabla_t \times \boldsymbol{h}_t = \sigma e_z \hat{z}, \quad (5)$$

$$-jk_z \hat{z} \times \boldsymbol{h}_t + \nabla_t h_z \times \hat{z} = \sigma \boldsymbol{e}_t, \quad (6)$$

where $\nabla_t = \hat{x}\partial/\partial x + \hat{y}\partial/\partial y$. Cross-multiplying (6) by $\hat{z}$ and substituting in (4) one gets

$$\nabla_t e_z \times \hat{z} = -j\omega\mu\left(1 - j\frac{k_z^2}{\omega\mu\sigma}\right)\boldsymbol{h}_t + j\frac{k_z}{\sigma}\nabla_t h_z. \quad (7)$$

With the assumption that the cross-sectional dimensions of the line are small compared to the wavelength of interest, the longitudinal component of the current is the dominant one. Thus, $\boldsymbol{e}_t$ is negligible compared to e_z, and (5) and (7) are easily identified as the ones governing the electromagnetic field behavior inside the conductors. With k_z written in terms of its real part β (phase constant) and imaginary part α (attenuation constant), $k_z = -j\alpha + \beta$, (7) takes the form

$$\nabla_t e_z \times \hat{z} = -j\omega\mu\left[\left(1 - \frac{2\alpha\beta}{\omega\mu\sigma}\right) - j\frac{\beta^2 - \alpha^2}{\omega\mu\sigma}\right]\boldsymbol{h}_t$$

$$+ j\frac{\beta - j\alpha}{\sigma}\nabla_t h_z. \quad (8)$$

Equation (8) can be simplified even further by recognizing that the assumption that displacement effects inside the conductors are negligible compared to the conduction effects can be expressed quantitatively as $\sigma \gg \omega\epsilon_{\text{eff}}$, where ϵ_{eff} is the effective dielectric constant for the microstrip structure, defined through the relation $\beta^2 = \omega^2\mu\epsilon_{\text{eff}}$. Obviously, this condition is satisfied for most practical applications of microstrip structures. This inequality can also be written in the form $\omega\mu\sigma \gg \beta^2$. This, then, along with the obvious inequality $\alpha < \beta$, shows that the term in brackets in (8) can be set to 1 for the frequencies of interest, and (8) takes the simpler form

$$\nabla_t e_z \times \hat{z} = -j\omega\mu \boldsymbol{h}_t + j\frac{\beta - j\alpha}{\sigma}\nabla_t h_z. \quad (9)$$

Actually, the propagation constant can be eliminated from (9) by invoking the relation $\nabla \cdot \boldsymbol{H} = 0$ for the homogeneous conductor. However, this is not required for the following development. Equations (5) and (9) can be combined to derive the equation for e_z inside the conductor. Taking the curl of (9) and using (5) it is straightforward to show that $e_z(x, y)$ satisfies

$$\nabla_t^2 e_z = j\omega\mu\sigma e_z. \quad (10)$$

III. Discussion

It is apparent that (10) is independent of the propagation constant k_z. Its solution inside the conductor requires the knowledge of either e_z or its normal derivative on the periphery of the conductor cross-section. Once the equation is solved, the current distribution inside the conductors and the associated conductor losses can be estimated. Obviously, the boundary values of e_z or its normal derivative on the periphery of the conductor are dependent on the physics of the fields in the exterior. It is now apparent that, under the assumptions made thus far, the exp $(-jk_z z)$ dependence of the fields can affect the distribution of the current density over the conductor cross-section only through these boundary values of either e_z or its normal derivative.

At this point, it is appropriate to distinguish between quasi-TEM, and non-TEM conditions. Under quasi-TEM conditions, the exterior electric and magnetic fields are approximately transverse to the direction of propagation and are calculated from the solution of an electrostatic and a magnetostatic problem, respectively. The appropriate boundary condition for the solution of (10) inside the conductors is then the normal derivative of e_z on the conductor boundary which is proportional to the tangential component of $\boldsymbol{h}_t$. Since $\boldsymbol{h}_t$ is obtained from the solution of a magnetostatic problem, its value is independent of the dielectric properties of the exterior region. Under these conditions, the methods used in [1]–[5] for conductor loss calculations are well justified and yield very accurate results. The only concern one might have is whether a sizable axial electric field component in a high-loss line may invalidate the quasi-TEM mode approximation. However, the analysis in [8] shows that the ratio $|e_z|/|\boldsymbol{e}_t|$ is small enough even for high-loss lines and the quasi-TEM mode assumption remains valid. At this point, it is appropriate to recall that quasi-TEM conditions hold for frequencies such that the cross-sectional dimensions of the line are small compared to the wavelength in the dielectric medium.

At higher frequencies, where the quasi-TEM conditions are violated due to geometric dispersion, (10) will still hold for as long as $\sigma \gg \omega\epsilon_{\text{eff}}$. Assuming that the longitudinal current is still the main source of conductor loss, a rigorous solution to the problem will require solution of (10) in the interior of the conductors, solution of Maxwell's equations in the exterior, and coupling of the two solutions through boundary conditions at the conductor boundaries. For such cases, the tangential component of $\boldsymbol{h}_t$ on the conductor boundary depends on the propagation constant, and the accuracy of the methods in [1]–[5] becomes questionable. On the other hand, this rigorous eigenvalue problem can become computationally intensive, especially if multilayered microstrip structures with multiple layers of metallization are involved. In addition, the nonrectangular shape of the strip conductors, which tends to be the rule rather than the exception in interconnection structures for high-speed, high-performance, VLSI/ULSI systems, further complicates matters since rigorous formulations like the one in [6] no longer apply.

However, if the conductor losses are small enough to be considered as a perturbation, the following approximate procedure may be used instead to predict the attenuation constant due to conductor losses. First, the exterior problem is solved assuming perfect conductors of finite thickness. This is an eigenvalue problem and its solution provides the propagation constant β and the attenuation constant α_d due to any losses in the dielectric. Both integral equation [9] and finite element methods [10] have been presented for the solution of this eigenvalue problem for microstrips with conductors of arbitrary cross sections. Losses in the ground plane or other shielding boundaries can also be accounted for through the use of a complex surface impedance [11]. From the solution of this eigenvalue problem, the value of the tangential $\boldsymbol{h}_t$ along the periphery of the conductor is obtained also. This, then, constitutes the necessary boundary condition for the solution of (10) in the interior of the conductors and the prediction of the current distribution and the per unit length time-averaged ohmic loss P_c. The associated attenuation constant a_c is then found as $a_c \approx P_c/2P_t$, where P_t is the time-averaged power flow along the line.

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